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On covering numbers of sublevel sets of analytic functions

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Abstract

In this paper we estimate covering numbers of sublevel sets of families of analytic functions depending analytically on a parameter. We use these estimates to study the local behavior of these families restricted to certain fractal subsets of $\mathbb{R}^N$.

Keywords: Covering numbers; Family of holomorphic functions; Sublevel set; Remez type inequality

1. Introduction

1.1

The main result of this paper estimates covering numbers of sublevel sets of families of analytic functions depending analytically on a parameter. Using these estimates we prove strong Remez type inequalities for the restrictions of analytic functions to certain fractal sets. The existence of such inequalities was conjectured in [4] in connection with the study of traces of Morrey–Campananto spaces to Markov subsets of $\mathbb{R}^N$. Motivated by boundary value problems for PDEs, classical trace theorems characterize traces of spaces of generalized smoothness (e.g., Sobolev, Besov etc.) to smooth submanifolds of $\mathbb{R}^N$. But in many cases one needs similar results for subsets of a more complicated geometric structure (for instance, after the change of variables initial data may be situated on a Lipschitz surface). The general project of the authors of [4] is devoted to the characterization of traces of spaces of a given generalized smoothness to (polynomially) regular subsets of $\mathbb{R}^N$ via local polynomial approximation. The Remez type

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inequalities are among the main tools of this approach. The results presented in [4] deal with the technically simplest case of Morrey–Campanato spaces (more general results will appear in the forthcoming book [5]).

1.2

To formulate the results of the present paper we introduce some notation.

Let us recall that for a metric space M and a relatively compact subset $S \subset M$ the *covering number* $Cov(S;\epsilon)$ is defined by

$$Cov(S;\epsilon) := \inf\left[card\left\{\{m_i\} \subset S;\, S \subset \bigcup_i B_\epsilon(m_i)\right\}\right] \tag{1.1}$$

where $B_r(m) \subset M$ is an open ball of radius r with center m. For $S = \emptyset$ we define $Cov(S;\cdot) = 0$.

Covering numbers are qualitative characteristics of the approximation of compact sets by means of n-point subsets; see, e.g., [14]. It is well known that Cov is a countably subadditive function in S and a nonincreasing and continuous from the right function in ϵ. For $M = \mathbb{R}^n$ we define the function Cov with respect to coverings by open l_∞ balls (cubes) which we denote by $K_r^n(x)$ (here x is the center of the cube and r is half of its side length).

Let $F \in \mathcal{O}(U \times V)$ be a holomorphic function on the product of the domains $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$. Then F can be considered as a family of holomorphic functions on V depending holomorphically on a parameter varying in U. The basic example of such a family is $\mathcal{P}_k(\mathbb{C}^n)$, the complex vector space of holomorphic polynomials of degree k on $\mathbb{C}^n$. In this case $U = \mathbb{C}^m$ where $m = d_{k,n} := \dim_{\mathbb{C}} \mathcal{P}_k(\mathbb{C}^n)$ and $V = \mathbb{C}^n$ and the corresponding function $F \in \mathcal{O}(U \times V)$ is a holomorphic polynomial of two variables such that $F(u,\cdot) \in \mathcal{P}_k(\mathbb{C}^n)$ and u is the vector of coefficients of $F(u,\cdot)$ ordered lexicographically.

We estimate covering numbers of sublevel sets of functions $F(u,\cdot) \in \mathcal{O}(V)$, $u \in U$, using two functions n_F and d_F; the first of these is defined by intersections of complex lines in $\mathbb{C}^n$ with sets of zeros of functions $F(u,\cdot)$ and the second one is closely related to the valencies of functions $F(u,\cdot)$ restricted to these lines.

Specifically, for each $u \in U$ and complex line $\ell \subset \mathbb{C}^n$ by $Z_F(u;\ell) \subset \ell$ we denote the set of zeros of the univariate holomorphic function $F(u,\cdot)|_{V\cap\ell}$. If $F(u,\cdot)|_{V\cap\ell} = 0$, we define $Z_F(u;\ell) := \emptyset$.

Let $X_U \subset\subset U$ and $\widetilde{X}_V \subset\subset V$ be relatively compact open sets. We set for $u \in X_U$

$$n_F(u;\widetilde{X}_V) := \max\left\{1;\, \sup_\ell \left\{card\{Z_F(u;\ell) \cap \widetilde{X}_V\}\right\}\right\} \tag{1.2}$$

where the supremum is taken over all complex lines ℓ in $\mathbb{C}^n$.

It is known (see, e.g., [10]) that

$$n_F(X_U,\widetilde{X}_V) := \sup_{u\in X_U} n_F(u;\widetilde{X}_V) < \infty. \tag{1.3}$$

To define d_F we use results of [1] asserting, in particular, that for every $u \in X_U$ there exists a number $d > 0$ such that for any real line $\ell_r \subset \mathbb{C}^n$ ($\cong\mathbb{R}^{2n}$) intersecting $\widetilde{X}_V$, open interval $I \subset \ell_r \cap \widetilde{X}_V$ and measurable subset ω of I the following Remez type inequality is true:

$$\sup_I |F(u,\cdot)| \le \left(\frac{4|I|}{|\omega|}\right)^d \sup_\omega |F(u,\cdot)|. \tag{1.4}$$

Here $|\cdot|$ is the Lebesgue measure on ℓ.

The optimal constant d in (1.4) denoted by $d_F(u; \widetilde{X}_V)$ is called the *Chebyshev degree* of $F(u, \cdot)$ with respect to $\widetilde{X}_V$.

As a corollary of (1.4) one obtains the following *Yu. Brudnyi–Ganzburg type inequality*; see [1, Theorem 1.9]:

Let S be a compact convex subset of $\widetilde{X}_V$ of dimension k and $\omega \subset S$ be a measurable subset. Then for every $u \in X_U$ the following inequality holds:

$$\sup_S |F(u, \cdot)| \le \left(\frac{4k\mathcal{L}_k(S)}{\mathcal{L}_k(\omega)}\right)^{d_F(u;\widetilde{X}_V)} \sup_\omega |F(u, \cdot)|. \tag{1.5}$$

Here $\mathcal{L}_k$ is the Lebesgue k-measure on the affine subspace of $\mathbb{C}^n$ ($\cong\mathbb{R}^{2n}$) generated by S.

Further, recall that a univariate holomorphic function defined on a bounded domain in $\mathbb{C}$ is *p-valent* if it assumes no value more than p times there. Let $W \subset\subset V$ be a relatively compact open subset containing the closure of $\widetilde{X}_V$. We define the valency of the family F on $X_U \times W$ by the formula

$$v_F(X_U, W) := \sup_{u\in X_U}\left\{\sup_\ell \{\text{valency of } F(u, \cdot)|_{W\cap\ell}\}\right\} \tag{1.6}$$

where the inner supremum is taken over all complex lines ℓ in $\mathbb{C}^n$.

The latter is known to be finite by results of [10]. (The proof is based on a resolution of singularities technique.) Using [1, Proposition 1.7] one obtains that there exists a constant c depending on X_U and W such that

$$d_F(X_U, \widetilde{X}_V) := \sup_{u\in X_U} d_F(u; \widetilde{X}_V) \le c v_F(X_U, W). \tag{1.7}$$

This, in particular, implies that the function $u \mapsto d_F(u; \widetilde{X}_V)$, $u \in X_U$, is finite.

Example 1.1. (1) For the family $F = \mathcal{P}_k(\mathbb{C}^n)$ of holomorphic polynomials of degree k on $\mathbb{C}^n$ with $U = \mathbb{C}^{d_{k,n}}$, $V = \mathbb{C}^n$ and $X_U := K_1^{2d_{k,n}}(0) \subset \mathbb{C}^{d_{k,n}}$, $\widetilde{X}_V := K_1^{2n}(0) \subset \mathbb{C}^n$ we clearly have $n_F(X_U, \widetilde{X}_V) = \max\{1; k\}$. Also, the Remez polynomial inequality implies that $d_F(X_U, \widetilde{X}_V) = k$.

(2) Let $l_1, \ldots, l_s \in (\mathbb{C}^n)^*$ be complex linear functionals. An exponential polynomial with spectrum $l_1, \ldots, l_s$ is a finite sum

$$f(z) = \sum_{i=1}^{s} p_i(z)e^{l_i(z)} \tag{1.8}$$

where the p_i are holomorphic polynomials on $\mathbb{C}^n$.

Let $F \in \mathcal{O}(U \times V)$ where $U := \mathbb{C}^m$, $m := s(d_{k,n}+n)$, and $V := \mathbb{C}^n$ be such that each $F(u, \cdot)$ is an exponential polynomial of the form (1.8) with all $p_i \in \mathcal{P}_k(\mathbb{C}^n)$ and u is the vector consisting of coefficients of all polynomials p_i and l_i in (1.8) for $F(u, \cdot)$. We set $X_U := K_1^{2m}(0) \subset \mathbb{C}^m$ and $\widetilde{X}_V := B_1(0)$, the Euclidean ball in $\mathbb{C}^n$ centered at 0 of radius 1. Then [1, Proposition 1.4] (see also the arguments in its proof) implies that there exists a numerical constant $c > 0$ such that

$$\max\left\{n_F(X_U, \widetilde{X}_V), d_F(X_U, \widetilde{X}_V)\right\} \le c(\sqrt{s}M + s(d_{k,n} + 1))$$

where $M := \max_{1\le i\le s}\{\|f_i\|_{l_2(\mathbb{C}^n)}\}$.

Next we fix a relatively compact domain $X_V \subset\subset \widetilde{X}_V$ and for $u \in X_U$, $S \subset X_V$ set

$$L_{F(u,\cdot)}(c) := \{z \in X_V;\ |F(u,z)| < c\}, \qquad M_{F(u,\cdot)}(S) := \sup_{z\in S} |F(u,z)|. \tag{1.9}$$

Our main result estimates covering numbers of the sets $L_{F(u,\cdot)}(c)$ in two cases:

(a) for X_V being a relatively compact open subset of $\widetilde{X}_V \subset\subset \mathbb{C}^n$ (in this case each $F(u,\cdot)|_{X_V}$, $u \in X_U$, is a holomorphic function);
(b) for X_V being a relatively compact open subset of $\widetilde{X}_V \cap \mathbb{R}^n$ (in this case each $F(u,\cdot)|_{X_V}$, $u \in X_U$, is a complex-valued analytic function).
Here we assume that $\widetilde{X}_V \cap \mathbb{R}^n \neq \emptyset$.

In particular, the first case can be applied to holomorphic polynomials on $\mathbb{C}^n$ while the second can be applied to complex polynomials on $\mathbb{R}^n$.

Convention. For $u \in X_U$ by $(X_V, d_u, 2^s n)$ we denote one of the triples: $(X_V, n_F(u; \widetilde{X}_V), 2n)$ with X_V as in (a) (here $s = 1$) or $(X_V, d_F(u; \widetilde{X}_V), n)$ with X_V as in (b) (here $s = 0$).

Theorem 1.2. *For a triple $(X_V, d_u, 2^s n)$, $u \in X_U$, there exists a positive constant $C = C(F, X_U, X_V, \widetilde{X}_V)$*[1] *such that for each $0 < \epsilon \le 1$ and $0 < t \le \epsilon$*

$$Cov\left(L_{F(u,\cdot)}\left(M_{F(u,\cdot)}(X_V)\cdot \epsilon^{d_u}\right); t\right) \le C \cdot \epsilon^{2^s} \cdot t^{-2^s n}. \tag{1.10}$$

Remark 1.3. Since $\epsilon \le 1$, inequality (1.10) is still valid if we replace d_u by a larger number. For instance, as such a number we can take $n_F(X_U, \widetilde{X}_V)$ in case (a) or $d_F(X_U, \widetilde{X}_V)$ in case (b).

Similar estimates are used in some problems of nonlinear approximation theory; see, e.g., [15,16] and references therein.

The proof of Theorem 1.2 in case (a) is based on Cartan type estimates for univariate holomorphic functions and in some cases gives an effective estimate of the constant C; see Theorem 3.1. The proof of Theorem 1.2 in case (b) is based on Yu. Brudnyi–Ganzburg type inequalities for analytic functions along with Vitushkin's inequality for covering numbers and Gabrielov's finiteness theorem for the number of connected components of compact semi-analytic sets. For certain families $\mathcal{F}$ one obtains an effective estimate of the constant C as well; see Example 4.5.

Remark 1.4. Originally the Vitushkin inequality was used by Yomdin [20] to prove Remez type inequalities for restrictions of real polynomials to certain finite subsets of $\mathbb{R}^N$; cf. Corollary 1.8.

As a corollary of Theorem 1.2 we obtain

Corollary 1.5. *Under the conditions of Theorem 1.2*

$$\mathcal{L}_{2^s n}\left(L_{F(u,\cdot)}\left(M_{F(u,\cdot)}(X_V)\cdot \epsilon^{d_u}\right)\right) \le C \cdot \epsilon^{2^s}.$$

Here $\mathcal{L}_N$ is the Lebesgue N-measure on $\mathbb{R}^N$.

[1] Here and below by $C = C(\alpha, \beta, \gamma, \ldots)$ we denote a constant depending on $\alpha, \beta, \gamma, \ldots$.

Remark 1.6. Some effective estimates of volumes of sublevel sets of certain classes of holomorphic polynomials and plurisubharmonic functions of the form similar to that of Corollary 1.5 were previously obtained in the papers [9,18,12,21].

Let us first show that inequality (1.10) is sharp in order.

Example 1.7. (1) Consider the family $F \in \mathcal{O}(U \times V)$ representing $\mathcal{P}_k(\mathbb{C}^n)$ with $k \geq 1$ where $U := \mathbb{C}^{d_{k,n}}$, $V := \mathbb{C}^n$. Also, we define $X_V := \mathbb{D}^n$, $X_U := \mathbb{D}^{d_{k,n}}$ where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk, and as the set $\widetilde{X}_V$ we choose a bounded domain containing the closure of X_V. Clearly, $d_u := n_F(u; \widetilde{X}_V) \leq k$. Therefore inequality (1.10) implies

$$Cov\left(L_{F(u,\cdot)}\left(M_{F(u,\cdot)}(\mathbb{D}^n) \cdot \epsilon^k\right); t\right) \leq C \cdot \epsilon^2 \cdot t^{-2n}.$$

On the other hand, consider $F(u_0, z) := 2^{-k} z_1^k \in \mathcal{P}_k(\mathbb{C}^n)$ where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Clearly, $u_0 = (2^{-k}, 0, \ldots, 0) \in X_U$, $M_{F(u_0,\cdot)}(X_V) = 2^{-k}$ and $d_{u_0} = k$. Therefore

$$L_{F(u_0,\cdot)}\left(M_{F(u_0,\cdot)}(X_V) \cdot \epsilon^{d_{u_0}}\right) = \{z \in \mathbb{D}^n;\ |z_1|^k < (2\epsilon)^k\} = \mathbb{D}_{2\epsilon} \times \mathbb{D}^{n-1}$$

where $\mathbb{D}_{2\epsilon}$ is the open disk of radius 2ϵ centered at 0.

A direct computation shows that there exists a constant $c = c(n)$ such that for every $0 < t \leq \epsilon \leq 1$

$$Cov(\mathbb{D}_{2\epsilon} \times \mathbb{D}^{n-1}; t) \geq c \cdot \epsilon^2 \cdot t^{-2n}.$$

Hence,

$$\sup_{u \in X_U} \left\{Cov\left(L_{F(u,\cdot)}\left(M_{F(u,\cdot)}(X_V) \cdot \epsilon^{d_u}\right); t\right)\right\} \sim \epsilon^2 \cdot t^{-2n}$$

with the constants of equivalence depending on F.

(2) For the same family now we set $X_V := K_1^n(0) \subset \mathbb{R}^n$. Then the Remez polynomial inequality implies that $d_u := d_F(u; \widetilde{X}_V) \leq k$. Thus from inequality (1.10) we obtain

$$Cov\left(L_{F(u,\cdot)}\left(M_{F(u,\cdot)}(X_V) \cdot \epsilon^k\right); t\right) \leq C \cdot \epsilon \cdot t^{-n}.$$

As before consider $F(u_0, z) := 2^{-k} z_1^k \in \mathcal{P}_k(\mathbb{C}^n)$. Then $M_{F(u_0,\cdot)}(X_V) = 2^{-k}$, $d_{u_0} = k$ and

$$L_{F(u_0,\cdot)}\left(M_{F(u_0,\cdot)}(X_V) \cdot \epsilon^k\right) = \{z \in K_1^n(0);\ |z_1|^k < (2\epsilon)^k\} = (-2\epsilon, 2\epsilon) \times K_1^{n-1}(0).$$

Clearly for every $0 < t \leq \epsilon \leq 1$

$$Cov\left((-2\epsilon, 2\epsilon) \times K_1^{n-1}(0); t\right) > \frac{1}{2^n} \cdot \epsilon \cdot t^{-n}.$$

Hence we obtain for $0 < \epsilon \leq 1$, $0 \leq t \leq \epsilon$

$$\sup_{u \in X_U} \left\{Cov\left(S_{F(u,\cdot)}\left(M_{F(u,\cdot)}(X_V) \cdot \epsilon^{d_u}\right); t\right)\right\} \sim \epsilon \cdot t^{-n}$$

with the constants of equivalence depending on F.

A subset $S \subset \mathbb{R}^N$ is said to be ϵ-*separated* if $\|s_1 - s_2\|_\infty \geq \epsilon$ for all pairs of distinct points $s_1, s_2 \in S$; here $\|\cdot\|_\infty$ is the l_∞ norm on $\mathbb{R}^N$.

As another corollary of Theorem 1.2 we obtain Remez type inequalities for functions from the family F.

Corollary 1.8. *For a triple $(X_V, d_u, 2^s n)$, $u \in X_U$, assume that a 2ϵ-separated set S with $0 < \epsilon \le 1$ belongs to X_V and $\mathrm{card}\, S > C \cdot \epsilon^{2^s(1-n)}$ where C is the constant in (1.10). Then*

$$\sup_{z \in X_V} |F(u,z)| \le \epsilon^{-d_u} \cdot \sup_{z \in S} |F(u,z)|. \tag{1.11}$$

2. Applications

Let $S \subset \mathbb{C}^n$ be a relatively compact subset such that $0 < \mu_\psi(S) < \infty$ where μ_ψ is the Carathéodory measure with weight $\psi : \mathbb{R}_+ \to \mathbb{R}_+$; see, e.g., [19] for the Carathéodory construction of the measure. In particular, if $\psi(t) = t^h$ with $h \in (0, 2n]$, then μ_ψ is the Hausdorff h-measure on $\mathbb{C}^n$.

The set S is said to be *μ_ψ-regular* if there is a constant c such that for every point $x \in S$ and every ball $K_r(x; S) := K_r(x) \cap S$ centered at this point of radius r

$$\mu_\psi(K_r(x; S)) \le c \cdot \psi(r). \tag{2.1}$$

It is well known that every measure space S of finite positive measure μ_ψ can be represented as the union of an increasing sequence of μ_ψ-regular subsets and a subset of μ_ψ-measure 0. In the forthcoming results we will assume that S itself is μ_ψ-regular with a constant c. The class of such an S will be denoted by $\mathcal{R}(\psi, c)$.

Next, for a triple $(X_V, d_u, 2^s n)$, $u \in X_U$, suppose that:

(*) $\psi(t) := t^{2^s(n-1)} \cdot \phi(t)$ where ϕ is a continuous increasing function on $\mathbb{R}_+$ such that $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$ and $\phi(t) \ge t^{2^s}$ for all $0 < t \le 1$.

Theorem 2.1. *If a set $S \in \mathcal{R}(\psi, c)$ belongs to X_V, then there exist positive constants $C_1 \le 1$ and $C_2 \ge 1$ depending on $F, X_U, X_V, \widetilde{X}_V$ such that for each ball $K_r(x; S)$, ψ-measurable subset $\omega \subset K_r(x; S)$ and $u \in X_U$*

$$\sup_{z \in K_r(x;S)} |F(u,z)| \le \left[\frac{C_1}{2r} \phi^{-1}\left(\frac{\mu_\psi(\omega)}{C_2 c r^{2^s(n-1)}}\right)\right]^{-d_u} \sup_{z \in \omega} |F(u,z)|. \tag{2.2}$$

As a consequence of the theorem one obtains the following integral inequalities.

Corollary 2.2. *Under the conditions of Theorem 2.1*

(1)

$$\sup_{z \in K_r(x;S)} |F(u,z)| \cdot \left(\int_0^1 \left[\frac{C_1}{2r} \phi^{-1}\left(\frac{\mu_\psi(K_r(x;S))t}{C_2 c r^{2^s(n-1)}}\right)\right]^{d_u} \mathrm{d}t\right) \le \frac{1}{\mu_\psi(K_r(x;S))} \int_{K_r(x;S)} |F(u,\cdot)| \mathrm{d}\mu_\psi.$$

(2) *If $F(u,\cdot) \not\equiv 0$, then*

$$d_u \cdot \left(\int_0^1 \log\left[\frac{C_1}{2r} \phi^{-1}\left(\frac{\mu_\psi(K_r(x;S))t}{C_2 c r^{2^s(n-1)}}\right)\right] \mathrm{d}t\right) \le \frac{1}{\mu_\psi(K_r(x;S))} \int_{K_r(x;S)} \log\left[\frac{|F(u,\cdot)|}{\sup_{z \in K_r(x;S)} |F(u,z)|}\right] \mathrm{d}\mu_\psi.$$

A set $S \in \mathcal{R}(\psi, c)$ is called a ψ-*set* (or a Cantor type set) if there exists a positive constant $\tilde{c}$ such that for each $K_r(x; S)$

$$\mu_\psi(K_r(x; S)) \geq \tilde{c}\psi(r). \tag{2.3}$$

The class of such S will be denoted by $\mathcal{R}(\psi, c, \tilde{c})$. It is known that a self-similar subset of $\mathbb{C}^n$ ($\cong\mathbb{R}^{2n}$) belongs to $\mathcal{R}(\psi, c, \tilde{c})$ where $\psi(t) = t^h$ for some $h > 0$ and $c, \tilde{c}$. This and other examples of Cantor type sets for Carathéodory measures (with not necessarily power functions ψ) can be found, e.g., in ([7], Chapter 4, [19]).

Now, under the conditions of Theorem 2.1 for $S \in \mathcal{R}(\psi, c, \tilde{c})$ from inequality (2.2) one obtains

$$\sup_{z\in K_r(x;S)} |F(u, z)| \leq \left[\frac{C_1}{2r}\phi^{-1}\left(\frac{\tilde{c}\mu_\psi(\omega)\phi(r)}{C_2 c\mu_\psi(K_r(x; S))}\right)\right]^{-d_u} \sup_{z\in\omega}|F(u, z)|. \tag{2.4}$$

Suppose that ϕ is such that:

(**) there exists a number $\alpha \in (0, 2^s]$ such that $\frac{\phi(t)}{t^\alpha}$ is nondecreasing on $\mathbb{R}_+$.

Since $C_2 \geq 1$,

$$A := \frac{\tilde{c}\mu_\psi(\omega)}{C_2 c\mu_\psi(K_r(x; S))} \leq 1.$$

In particular, for such a ϕ we obtain

$$\frac{\phi(r)}{r^\alpha} \geq \frac{\phi(A^{1/\alpha}r)}{(A^{1/\alpha}r)^\alpha}$$

implying that

$$\frac{1}{2r}\phi^{-1}(A\phi(r)) \geq \frac{A^{1/\alpha}}{2}.$$

From here and (2.4) we have for every $u \in X_U$

$$\sup_{z\in K_r(x;S)} |F(u, z)| \leq \left[\frac{2^\alpha C_2 c\mu_\psi(K_r(x; S))}{C_1^\alpha\tilde{c}\mu_\psi(\omega)}\right]^{\frac{d_u}{\alpha}} \sup_{z\in\omega}|F(u, z)|. \tag{2.5}$$

An inequality of the form (2.5), i.e., where the constant depends on the ratio of the corresponding measures polynomially, is called in [4] a *strong Remez type inequality*. In turn, if the corresponding constant depends on the ratio of measures implicitly, then the corresponding inequality is called in [4] a *weak Remez type inequality*. Such inequalities are important for extension and trace problems for certain spaces of smooth functions on $\mathbb{R}^N$; see, e.g., [4].

It was established in [6,4] that the family of holomorphic polynomials $\mathcal{P}_k(\mathbb{C}^n)$ satisfies a strong Remez type inequality on a ψ-set $S \subset \mathbb{C}^n$ for μ_ψ being the Hausdorff h-measure on $\mathbb{C}^n$ with $h \in (2n-2, 2n]$, and the family $\mathcal{P}_k(\mathbb{C}^n)|_{\mathbb{R}^n}$ of complex polynomials on $\mathbb{R}^n$ satisfies a weak integral Remez inequality on a ψ-set $S \subset \mathbb{R}^n$ for μ_ψ being the Hausdorff h-measure on $\mathbb{R}^n$ with $h \in (n-1, n]$. The first case corresponds to (2.5) for $\mathcal{P}_k(\mathbb{C}^n)$ with X_V being an open ball in $\mathbb{C}^n$ and $\psi(t) = t^{2n-2}\phi(t)$ where $\phi(t) = t^l$ and $l \in (0, 2]$, and the second is an implicit form of inequality (1) of Corollary 2.2 with X_V being an open ball in $\mathbb{R}^n$ and $\psi(t) = t^{n-1}\phi(t)$ where $\phi(t) = t^l$ and $l \in (0, 1]$.

It was also conjectured in [4] that a strong Remez type inequality of the form (2.5) is valid for complex polynomials on $\mathbb{R}^n$ restricted to a compact ψ-set $S \subset \mathbb{R}^n$ with $\psi(t) = t^{n-1+l}$, $l \in (0, 1]$. The proof of this conjecture can be obtained from the main result of the recent paper of Yomdin [20] by the method of the proof of Theorem 2.1. In turn, Theorem 2.1 expressed in the form (2.5) proves this conjecture in a more general setting of holomorphic or analytic families of functions on relatively compact domains in $\mathbb{C}^n$ or $\mathbb{R}^n$, respectively, and weights ψ satisfying (*) and (**). (For instance, as such a weight one can take $\psi(t) := t^{2^s(n-1)}\phi(t)$ where $\phi(t) := t^l \cdot g(t)$, $l \in (0, 2^s)$, and $g(t) = \frac{1}{|\log t|^k}$, $k > 0$, for $0 < t \le \mathrm{e}^{-1}$, and $g(t) = 1$ for $t > \mathrm{e}^{-1}$.)

The restriction for S of having ψ-dimension greater than $2^s(n-1)$ is explained by the fact that the topological dimension of the set of zeros of a generic holomorphic function on a domain of $\mathbb{C}^n$ is $2n-2$ and the topological dimension of a generic analytic function on a domain of $\mathbb{R}^n$ is $n-1$.

From (2.5) for $S \in \mathcal{R}(\psi, c, \tilde{c})$ and ψ satisfying (*) and (**) as in the proof of Corollary 2.2 we obtain the following integral inequalities for F satisfying the conditions of Theorem 2.1.

Corollary 2.3. (1)

$$\sup_{z \in K_r(x;S)} |F(u,z)| \cdot \left[\frac{\alpha}{d_u + \alpha}\right] \cdot \left[\frac{C_1^\alpha \tilde{c}}{C_2 c 2^\alpha}\right]^{\frac{d_u}{\alpha}} \le \frac{1}{\mu_\psi(K_r(x;S))} \int_{K_r(x;S)} |F(u,\cdot)| \mathrm{d}\mu_\psi.$$

(2) *If* $F(u,\cdot) \not\equiv 0$, *then*

$$\left[\frac{d_u}{\alpha}\right] \cdot \log\left[\frac{C_1^\alpha \tilde{c}}{eC_2 c 2^\alpha}\right] \le \frac{1}{\mu_\psi(K_r(x;S))} \int_{K_r(x;S)} \log\left[\frac{|F(u,\cdot)|}{\sup_{z \in K_r(x;S)} |F(u,z)|}\right] \mathrm{d}\mu_\psi.$$

The first inequality of the corollary is the reverse Hölder inequality for functions from F. The second inequality shows that $(\log|F(u,\cdot)|)|_S$ belongs to $BMO(S)$ defined with respect to μ_ψ with the BMO norm bounded by $\left(\frac{2d_u}{\alpha}\right) \cdot \log\left(\frac{eC_2 c 2^\alpha}{C_1^\alpha \tilde{c}}\right)$. These results extend similar results for the families $\mathcal{P}_k(\mathbb{C}^n)$ of holomorphic polynomials on $\mathbb{C}^n$ established in [4]; see also the references therein.

3. Proof of Theorem 1.2 in case (a)

3.1

The proof of Theorem 1.2 in this case will be deduced from the main result of this subsection. In its formulation $B_R(0) \subset \mathbb{C}^n$ denotes the open Euclidean ball of radius R centered at the origin. By $\mathcal{O}_R$ we denote the space of holomorphic functions on $B_R(0)$. For $f \in \mathcal{O}_R$ and $R' < R$ we set

$$n_f(R') := \sup_\ell [card\{(Z_f \cap B_{R'}(0)) \cap \ell\}] \tag{3.1}$$

where Z_f is the set of zeros of f and the supremum is taken over all complex lines ℓ in $\mathbb{C}^n$ which do not belong to Z_f. According to the results of [10], $n_f(R') < \infty$. Also, we define

$$L_{f;R'}(c) := \{z \in B_{R'}(0); |f(z)| < c\} \quad \text{and} \quad M_f(R') := \sup_{B'_R(0)} |f|. \tag{3.2}$$

In the following result Cov_2 denotes the covering numbers function defined with respect to coverings by open Euclidean balls. Clearly, on $\mathbb{R}^N$ the functions Cov_2 and Cov are equivalent with the constants of equivalence depending only on N.

Theorem 3.1. *Assume that $f \in \mathcal{O}_R$ is not identically zero and $R_* < R$. For $R' := \frac{R_*+3R}{4}$, $R'' := \frac{R_*+R}{2}$ we set*

$$c_f(R_*) := (24e) \cdot \left(\frac{M_f(R')}{M_f(R_*)}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2 \cdot \frac{1}{n_f(R')}}. \tag{3.3}$$

Suppose that $n_f(R') \geq 1$. Then for every $0 < \epsilon \leq \frac{3(R-R_*)}{c_f(R_*)R'}$

$$\begin{aligned} &Cov_2\left(L_{f;R_*}(M_f(R_*) \cdot \epsilon^{n_f(R')}); \epsilon \cdot c_f(R_*) \cdot R'\right) \\ &\quad \leq n \cdot n_f(R') \cdot \left(\frac{c_f(R_*)}{12}\right)^{-2n+2} \cdot \epsilon^{-2n+2}. \end{aligned} \tag{3.4}$$

If $n_f(R') = 0$, then for any $\epsilon \leq \left(\frac{M_f(R_*)}{M_f(R')}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2}$

$$L_{f;R_*}(M_f(R_*) \cdot \epsilon) = \emptyset.$$

The next two subsections contain some results used in the proof of this theorem.

3.2

Let $\omega_e := \frac{\sqrt{-1}}{2} \sum_{1 \leq i \leq n} dz_i \wedge d\bar{z}_i$ be the Euclidean Kähler (1, 1)-form which determines the Euclidean metric on $\mathbb{C}^n$. Let $X \subset B_R(0) \subset \mathbb{C}^n$ be a complex analytic subset of pure (complex) dimension p. For $R' < R$ we set

$$n_X(R') := \sup_{\pi}[card\{(X \cap B_{R'}(0)) \cap \pi\}]$$

where the supremum is taken over all complex affine subspaces π of $\mathbb{C}^n$ of complex dimension $n - p$ that intersect $X \cap B_{R'}(0)$ in finitely many points.

By $\mu_{e,X}$ we denote the measure on X defined on Borel subsets $U \subset X$ by the formula

$$\mu_{e,X}(U) := \int_U \wedge^p \omega_e. \tag{3.5}$$

Then for $B_r(z; X) := B_r(z) \cap X$ such that $z \in X$ and $B_r(z) \subset B_{R'}(0)$ we have

$$c(p)r^{2p} \leq \mu_{e,X}(B_r(z; X)) \leq n_X(R') \binom{n}{p} c(p)r^{2p} \tag{3.6}$$

where $c(p) := \frac{\pi^p}{p!}$ is the volume of the unit Euclidean ball in $\mathbb{C}^p$; see, e.g., [8, Chapters III.3.4, III.3.5]. (The right-hand inequality in (3.6) is valid also for $z \notin X$.)

As a corollary of inequality (3.6) we obtain

Proposition 3.2. *Let $R'' < R'$ and $S \subset X \cap B_{R''}(0)$ be an ϵ-separated subset for $0 < \epsilon < 2(R' - R'')$, i.e., for all distinct points $s_i, s_j \in S$*

$$\|s_i - s_j\|_2 \geq \epsilon$$

(here $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{C}^n \cong \mathbb{R}^{2n}$). Then

$$card S \leq n_X(R') \binom{n}{p} \left(\frac{2R'}{\epsilon}\right)^{2p}. \tag{3.7}$$

Proof. By the definition of S, the balls $B_{\frac{\epsilon}{2}}(s_i; X)$, $s_i \in S$, are mutually disjoint and belong to $X \cap B^{2n}_{R'}(0)$. Thus according to (3.6)

$$\begin{aligned} card S \cdot c(p) \left(\frac{\epsilon}{2}\right)^{2p} &\leq \sum_{s_j \in S} \mu_{e,X}(B_{\frac{\epsilon}{2}}(s_j; X)) \\ &\leq \mu_{e,X}(X \cap B_{R'}(0)) \leq n_X(R') \binom{n}{p} c(p)(R')^{2p}. \end{aligned}$$

This implies the result. □

We set $dist(z, Y) := \inf_{y \in Y} \|z - y\|_2$.

Corollary 3.3. *Let $S \subset B_{R''}(0)$ be a 3ϵ-separated subset with $0 < \epsilon < 2(R' - R'')$. Assume that*

$$card S > n_X(R') \binom{n}{p} \left(\frac{2R'}{\epsilon}\right)^{2p}.$$

Then there is $s_* \in S$ such that

$$dist(s_*, X \cap B_{R''}(0)) \geq \epsilon.$$

Proof. Suppose, on the contrary, that $dist(s, X \cap B_{R''}(0)) < \epsilon$ for all $s \in S$. Then for each $s \in S$ there exists $d(s) \in X \cap B_{R''}(0)$ such that $\|s - d(s)\|_2 < \epsilon$ and, moreover, by the triangle inequality for distinct $s', s'' \in S$ we have $\|d(s') - d(s'')\|_2 \geq \epsilon$. In particular,

$$card\{d(s); s \in S\} = card(S) > n_X(R') \binom{n}{p} \left(\frac{2R'}{\epsilon}\right)^{2p}.$$

This contradicts Proposition 3.2. Hence, there exists a point $s_* \in S$ such that $dist(s_*, X \cap B_{R''}(0)) \geq \epsilon$. □

3.3

We also use Cartan type inequalities for univariate holomorphic functions; see, e.g., [2, Section 3.2].

Let f be a nonzero holomorphic function in the open disk $\mathbb{D}_R$ centered at 0 of radius R. Fix positive α, β such that $\alpha < \beta < 1$. By $n_f(r)$ we denote the number of zeros of f in the disk $\mathbb{D}_r$ and by $M_f(r)$ supremum of f in $\mathbb{D}_r$.

Theorem 3.4. *Let H be a positive number $\leq \beta e$. Then there exists a family of open disks $\{D_j\}_{1\leq j\leq k}$, $k \leq n_f(\beta R)$, with $\sum r_j \leq 2HR$ where r_j is the radius of D_j such that*

$$|f(z)| \geq M_f(\beta R)\left(\frac{M_f(\alpha R)}{M_f(\beta R)}\right)^{\left(\frac{\beta+\alpha}{\beta-\alpha}\right)^2} \cdot \left(\frac{H}{\beta e}\right)^{n_f(\beta R)}$$

for any $z \in \mathbb{D}_{\alpha R} \setminus \cup_j D_j$.

Moreover, the family $\cup_j D_j$ covers the set of zeros of f in $\mathbb{D}_{\beta R}$.

3.4. *Proof of* Theorem 3.1

Assume that a holomorphic function f is defined in the ball $B_R(0) \subset \mathbb{C}^n$. For a fixed $R_* < R$ we set $R'' := \frac{R_*+R}{2}$, $R' := \frac{R_*+3R}{4}$.

Consider a subset $S \subset B_{R_*}(0)$ satisfying the conditions:

(1) $\|s_i - s_j\|_2 \geq 3\delta$ for all distinct $s_i, s_j \in S$ with $0 < \delta \leq \frac{R-R_*}{2}$ $(:=R-R'')$;

(2)

$$card S > n_f(R') \cdot n \cdot \left(\frac{R_* + 3R}{2\delta}\right)^{2n-2}.$$

Since the set of zeros $Z_f \subset B_R(0)$ of f is either $\emptyset$ or a pure $(n-1)$-dimensional complex analytic subset of $B_R(0)$ and $n_f(R') := n_{Z_f}(R')$, from Corollary 3.3 and the above conditions we obtain that there exists a point $s_* \in S$ such that

$$dist(s_*, Z_f \cap B_{R''}(0)) \geq \delta. \tag{3.8}$$

Let us prove the following result.

Lemma 3.5. *s_* does not belong to the sublevel set $L_{f;R_*}\left(M_f(R_*) \cdot \kappa_f \cdot \delta^{n_f(R')}\right)$ with*

$$\kappa_f := \left(\frac{M_f(R_*)}{M_f(R')}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2} \cdot \left(\frac{1}{4eR'}\right)^{n_f(R')}. \tag{3.9}$$

Proof. Let $y \in \partial B_{R_*}(0)$ be a boundary point such that

$$|f(y)| = \sup_{z\in B_{R_*}(0)} |f| =: M_f(R_*) \tag{3.10}$$

and $\ell \subset \mathbb{C}^n$ be a complex line passing through y and s_*. We naturally identify ℓ with $\mathbb{C}$ and $\ell \cap B^{2n}_R(0)$, $\ell \cap B^{2n}_{R'}(0)$, $\ell \cap B^{2n}_{R''}(0)$ and $\ell \cap B^{2n}_{R_*}(0)$ with disks $\mathbb{D}_r$, $\mathbb{D}_{r'}$, $\mathbb{D}_{r''}$ and $\mathbb{D}_{r_*}$, respectively; here r, r', r'', r_* are radii of the disks obtained by intersection of the above balls with ℓ (for similar arguments see, e.g., part (1) of the proof of Theorem 3.4 of [2]). Then we have $r \leq R$, $r' \leq R'$, $r'' \leq R''$, $r_* \leq R_*$ and

$$r'' - r_* \geq R'' - R_*, \qquad r' - r'' \geq R' - R'', \qquad \frac{r''}{r'} \leq \frac{R''}{R'}, \qquad \frac{r'}{r} \leq \frac{R'}{R}. \tag{3.11}$$

We also set $\tilde{f} := f|_\ell$ and apply Theorem 3.4 to $\tilde{f}$ on $\mathbb{D}_r$ with $\alpha := \frac{r''}{r}$, $\beta := \frac{r'}{r}$ and $H := \frac{\delta}{4R}$ $(\le \frac{R-R_*}{8R} = \frac{R'-R''}{2R} \le \frac{r'-r''}{2r} < \beta e)$. Then there exists a family of open disks $\{D_j\}_{1\le j\le k}$ with $k \le n_{\tilde{f}}(r')$ and $\sum r_j \le \frac{\delta}{2}$ where r_j is the radius of D_j such that

$$|f(z)| \ge M_{\tilde{f}}(r')\left(\frac{M_{\tilde{f}}(r'')}{M_{\tilde{f}}(r')}\right)^{\left(\frac{r'+r''}{r'-r''}\right)^2} \cdot \left(\frac{r\delta}{4Rr'e}\right)^{n_{\tilde{f}}(r')}$$

$$\ge M_f(R_*)\left(\frac{M_f(R_*)}{M_f(R')}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2} \cdot \left(\frac{\delta}{4eR'}\right)^{n_f(R')} \tag{3.12}$$

for any $z \in \mathbb{D}_{r''} \setminus \cup_j D_j$.

We used here that $M_{\tilde{f}}(r') \le M_{\tilde{f}}(r_*) = M_f(R_*)$ (see (3.10)) and $M_{\tilde{f}}(r'') \ge M_{\tilde{f}}(r_*)$, $M_{\tilde{f}}(r') \le M_f(R')$, $\frac{r'+r''}{r'-r''} \le \frac{R'+R''}{R'-R''}$ (see (3.11)).

Also, each disk D_j contains a point of the set of zeros of $\tilde{f}$ in $\mathbb{D}_{r'}$.

Further, $s_* \in \mathbb{D}_{r_*}$ and the distance from s to the set of zeros of $\tilde{f}$ in $\mathbb{D}_{r''}$ is $\ge\delta$; see (3.8). Also, by the triangle inequality, the distance from s_* to the set of zeros of $\tilde{f}$ in $\mathbb{D}_{r'} \setminus \mathbb{D}_{r''}$ is $\ge r'' - r_* \ge R'' - R_* = \frac{R-R_*}{2} \ge \delta$. From here, since each D_j contains a point of the set of zeros of $\tilde{f}$ in $\mathbb{D}_{r'}$ and $\sum_j r_j \le \frac{\delta}{2}$, we obtain that $s_* \notin \cup_j D_j$. Thus $f(s_*)$ satisfies inequality (3.12). In turn, $s_* \notin L_{f;R_*}\left(M_f(R_*) \cdot \kappa_f \cdot \delta^{n_f(R')}\right)$. $\square$

Now let us prove Theorem 3.1.

Assume first that $Z_f(R') \ne \emptyset$. For an $\epsilon \in \left(0, \frac{3(R-R_*)}{c_f(R_*)R'}\right]$ with $c_f(R_*)$ defined by (3.3), let S be an $(\frac{\epsilon}{2} \cdot c_f(R_*) \cdot R')$-net (i.e., a maximal $(\frac{\epsilon}{2} \cdot c_f(R_*) \cdot R')$-separated subset) in $L_{f;R_*}(M_f(R_*) \cdot \epsilon^{n_f(R')})$. We set

$$\delta := \epsilon \cdot \kappa_f^{-1/n_f(R')}.$$

Then by the definitions of $c_f(R_*)$ and κ_f we obtain $\frac{\epsilon}{2} \cdot c_f(R_*) \cdot R' = 3\delta$. Now, Lemma 3.5 implies that

$$card S \le n_f(R') \cdot n \cdot \left(\frac{12}{c_f(R_*)\epsilon}\right)^{2n-2}.$$

Since open balls of radius $\epsilon \cdot c_f(R_*) \cdot R'$ centered at the points of S cover the sublevel set $L_{f;R_*}(M_f(R_*) \cdot \epsilon^{n_f(R')})$, the previous estimate gives the required inequality (3.4).

Suppose now that $Z_f(R') = \emptyset$. Then by Lemma 3.5 we obtain that

$$\inf_{z\in B_{R_*}(0)} |f(z)| \ge M_f(R_*)\left(\frac{M_f(R_*)}{M_f(R')}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2}.$$

This implies that $L_{f;R_*}(M_f(R_*) \cdot \epsilon) = \emptyset$ for any $\epsilon \le \left(\frac{M_f(R_*)}{M_f(R')}\right)^{\left(\frac{R'+R''}{R'-R''}\right)^2}$.

The proof of the theorem is complete. $\square$

3.5

As a corollary of Theorem 3.1 we obtain

Corollary 3.6. *Under the notation of Theorem 3.1 consider a family of holomorphic functions $F \in \mathcal{O}(U \times V)$ where $V = B_R(0) \subset \mathbb{C}^n$ and $U \subset \mathbb{C}^m$ is an open domain. Let $X_U \subset\subset U$ be a relatively compact domain. Then there exists a positive constant $C = C(F, X_U, V, n)$ such that for each $0 < \epsilon \le 1$, $0 < t \le \epsilon$, $u \in X_U$,*

$$Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot \epsilon^{\hat{n}_{F(u,\cdot)}(R')}); t\right) \le C \cdot \epsilon^2 \cdot t^{-2n};$$

here $\hat{n}_{F(u,\cdot)}(R') := \max\{1; n_{F(u,\cdot)}(R')\}$.

Proof. Using arguments of [1, Section 2] we obtain

$$\sup_{F(u,\cdot)\ne 0, u\in X_U} \frac{M_{F(u,\cdot)}(R')}{M_{F(u,\cdot)}(R_*)} =: M < \infty \tag{3.13}$$

and

$$\sup_{F(u,\cdot)\ne 0, u\in X_U} n_{F(u,\cdot)}(R') := N < \infty. \tag{3.14}$$

The first inequality implies that there exists a positive constant $c = c(M, R, R_*)$ such that for every $F(u, \cdot) \in \mathcal{O}_R$, $u \in X_U$, with $n_{F(u,\cdot)}(R') \ge 1$ the constant $c_{F(u,\cdot)}(R_*)$ defined by (3.3) is bounded from above by c.

Let us prove now the corollary for $t = \epsilon$. Assuming, first, that $n_{F(u,\cdot)}(R') \ge 1$, we obtain from (3.5) for all ϵ satisfying $0 < \epsilon \le \frac{3(R-R_*)}{cR'}\left(\le \frac{3(R-R_*)}{c_{F(u,\cdot)}(R_*)R'}\right)$

$$\begin{aligned} &Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot \epsilon^{n_{F(u,\cdot)}(R')}); \epsilon \cdot c_{F(u,\cdot)}(R_*)\cdot R'\right)\\ &\quad \le n\cdot N\cdot \left(\frac{c}{12}\right)^{-2n+2}\cdot \epsilon^{-2n+2}. \end{aligned} \tag{3.15}$$

If $c_{F(u,\cdot)}(R_*)\cdot R' \le 1$, then the same estimate is valid if we replace in (3.15) $\epsilon \cdot c_{F(u,\cdot)}(R_*)\cdot R'$ by ϵ. Otherwise, each ball of radius $\epsilon \cdot c_{F(u,\cdot)}(R_*)\cdot R'$ can be covered by at most $\gamma\cdot(c_{F(u,\cdot)}(R_*)\cdot R')^{2n}$ balls of radius ϵ for some γ depending on n. This and (3.15) yield for all ϵ satisfying $0 < \epsilon \le \frac{3(R-R_*)}{cR'}$

$$Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot \epsilon^{n_{F(u,\cdot)}(R')}); \epsilon\right) \le C_1 \cdot \epsilon^{-2n+2} \tag{3.16}$$

where C_1 depends on R, R_*, M and n.

If now $\frac{3(R-R_*)}{cR'} < \epsilon \le 1$, then $B_{R_*}(0)$ (and so $L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot \epsilon^{n_{F(u,\cdot)}(R')})$) can be covered by at most $\gamma\cdot\left(\frac{cR'\max\{1,R_*\}}{3(R-R_*)}\right)^{2n}$ balls of radius ϵ. This gives an estimate similar to (3.16) in this case and completes the proof of the corollary for $t = \epsilon$ and $n_{F(u,\cdot)}(R') \ge 1$.

Assume now that $n_{F(u,\cdot)}(R') = 0$. Then Theorem 3.1 and (3.13) imply that for any $0 < \epsilon \le c_1$ for some c_1 depending on M, R and R_*

$$L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot \epsilon) = \emptyset.$$

If now, $c_1 < \epsilon \le 1$, then $B_{R_*}(0)$ (and therefore $L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot\epsilon)$) can be covered by at most $\gamma\cdot\left(\frac{\max\{1,R_*\}}{c_1}\right)^{2n}$ balls of radius ϵ. This gives the required bound of $Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot\epsilon;\epsilon)\right)$ in this case.

The estimate for $Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot\epsilon^{\hat{n}_{F(u,\cdot)}(R')});t\right)$ with $0<t\le\epsilon$ follows from that for $Cov_2\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(R_*)\cdot\epsilon^{\hat{n}_{F(u,\cdot)}(R')});\epsilon\right)$ using the fact that each ball of radius ϵ can be covered by at most $\gamma\cdot\left(\frac{\epsilon}{t}\right)^{2n}$ balls of radius t.

The proof of the corollary is complete. □

3.6

Now, let us prove Theorem 1.2 in case (a).

First fix a finite cover $\mathcal{B}=\{B_i\}_{i\in I}$ of $X_V(\subset\subset\mathbb{C}^n)$ by open Euclidean balls such that $\cup_{i\in I}B_i$ is a relatively compact subset of $\widetilde{X}_V$. Since the Chebyshev degree $d_F(X_U,\widetilde{X}_V)$ is finite (see (1.4), (1.7)), repeating word-for-word the proof of [1, Theorem 1.9] in the complex case we obtain for each $u\in X_U$, and $B_i\cap B_j\neq\emptyset$

$$\sup_{B_i}|F(u,\cdot)|\le\left(\frac{8n\mathcal{L}_{2n}(B_i)}{\mathcal{L}_{2n}(B_i\cap B_j)}\right)^{d_F(X_U,\widetilde{X}_V)}\sup_{B_i\cap B_j}|F(u,\cdot)|. \tag{3.17}$$

Eq. (3.17) implies that there exist positive numbers c_{ij} depending on constants in (3.17) such that for each $u\in X_U$ and $B_i\cap B_j\neq\emptyset$

$$\frac{1}{c_{ij}}\cdot\sup_{B_j}|F(u,\cdot)|\le\sup_{B_i}|F(u,\cdot)|\le c_{ij}\cdot\sup_{B_j}|F(u,\cdot)|.$$

Since the cover $\mathcal{B}$ is finite, the latter implies that there are positive constants c_i depending on c_{ij} and the number of elements of the cover such that for each $u\in X_U$

$$\sup_{X_V}|F(u,\cdot)|\le c_i\cdot\sup_{B_i}|F(u,\cdot)|. \tag{3.18}$$

In the proof we will work with the function Cov_2 which is equivalent to Cov with the constants of equivalence depending on n only. Since Cov_2 is subadditive in the first argument and the cover $\mathcal{B}$ is finite, it suffices to estimate

$$Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u});t\right)$$

where $L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u}):=\{z\in B_i;|F(u,z)|<M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u}\}$, $0<\epsilon\le1$, $0<t\le\epsilon$, and $d_u:=n_F(u;\widetilde{X}_V)$, see (1.2). In turn, according to (3.18) it suffices to estimate

$$Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(B_i)\cdot c_i\cdot\epsilon^{d_u});t\right).$$

Further, we can consider only the case $t=\epsilon$ because for other t the required estimate can be obtained from this case as in the proof of Corollary 3.6.

By definition, there exists an open ball $\widetilde{B}_i$ with the same center as B_i such that $\widetilde{B}_i\subset\subset\widetilde{X}_U$. Fix R' larger than the radius of B_i but smaller than the radius of $\widetilde{B}_i$. Then from Corollary 3.6 we obtain for each $u\in X_U$ and $0<\epsilon\le1$

$$Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(B_i)\cdot\epsilon^{\hat{n}_{F(u,\cdot)}(R')});\epsilon\right)\le C_i\cdot\epsilon^{-2n+2}$$

where $C_i=C_i(F,X_U,\widetilde{B}_i,B_i,R',n)$.

By definition, $1 \le \hat{n}_{F(u,\cdot)}(R') \le n_F(u; \widetilde{X}_V)(=:d_u)$ and $\epsilon < 1$. Hence, the latter yields

$$Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(B_i) \cdot \epsilon^{d_u}); \epsilon\right) \le C_i \cdot \epsilon^{-2n+2}. \tag{3.19}$$

Now, for $\epsilon \le \left(\frac{1}{c_i}\right)^{\frac{1}{d_u}}$ (observe that $c_i \ge 1$) we have from here

$$\begin{aligned} &Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(B_i) \cdot c_i \cdot \epsilon^{d_u}); \epsilon\right) \\ &\quad \le \gamma \cdot (c_i)^{2n/d_u} \cdot Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(B_i) \cdot c_i \cdot \epsilon^{d_u}); (c_i)^{1/d_u} \cdot \epsilon\right) \le C_i' \cdot \epsilon^{-2n+2}. \end{aligned}$$

Here γ depends on n and C_i' on c_i, F and γ.

On the other hand, for $\epsilon > \left(\frac{1}{c_i}\right)^{\frac{1}{d_u}}$ the ball B_i can be covered by at most $\gamma \cdot (\max\{1, R_i\} \cdot (c_i)^{1/d_u})^{2n}$ balls of radius ϵ where R_i is the radius of B_i. This and the previous estimate give the required result:

There exists a constant $C = C(F, X_U, X_V, \widetilde{X}_V)$ such that for every $u \in X_U$, $0 < \epsilon \le 1$

$$\max_{i \in I}\left\{Cov_2\left(L^i_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V) \cdot \epsilon^{d_u}); \epsilon\right)\right\} \le C \cdot \epsilon^{-2n+2}.$$

The proof of Theorem 1.2 in case (a) is complete. □

4. Proof of Theorem 1.2 in case (b)

4.1

In this and following sections we gather some auxiliary results used in the proof.

We will require the Gabrielov finiteness theorem for the number of connected components of a family of compact semi-analytic sets depending analytically on a parameter; see [11, Corollary 1]. Let us recall the corresponding definition.

A subset $S \subset \mathbb{R}^n$ is called *semi-analytic* if in a neighbourhood of each point $x_0 \in \mathbb{R}^n$ it is the finite union of sets of the form $\{f_i = 0,\ g_j > 0,\ 1 \le i \le i_0,\ 1 \le j \le j_0\}$ where f_i and g_j are real analytic functions in the neighbourhood.

To formulate Gabrielov's result we introduce some notation.

By $I^n \subset \mathbb{R}^n$ we denote a closed cube. If $I^n := I^k \times I^{n-k}$, where $x' = (x_1, \dots, x_k)$ are the coordinates in I^k and $x'' = (x_{k+1}, \dots, x_n)$ are the coordinates in I^{n-k}, by $(x'')_{x_0'}$, $x_0' \in I^k$, we denote the subset $\{x_0'\} \times I^{n-k}$ in I^n. If S is a subset of I^n, by $S_{x_0'}(x'')$ we denote the set $S \cap (x'')_{x_0'}$. As a consequence of Corollary 1 of [11] one obtains:

Let $S := S(x, \xi) \subset I^{n+k}$ be a compact semi-analytic subset, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots \xi_k)$. Let $N(\xi)$ be the number of connected components of $S_\xi(x)$. Then

$$\sup_{\xi \in I^k} N(\xi) < \infty. \tag{4.1}$$

Example 4.1. If $S \subset \mathbb{R}^n$ is *semi-algebraic*, defined by polynomial inequalities $f_1 \ge 0, \dots, f_p \ge 0$, where f_j are real polynomials on $\mathbb{R}^n$, a bound on the number of connected components $N(S)$ of S is obtained by Milnor [17]:

$$N(S) \le \frac{1}{2}(2 + d)(1 + d)^{n-1} \tag{4.2}$$

where $d := deg f_1 + \cdots + deg f_p$.

4.2

We will use some results on variations of sets; see, e.g., the book [13] for the basic results and references.

Let $G \subset \mathbb{R}^n$ be an open subset. For a compact set $S \subset G$ by $V_0(S, G)$ we denote the number of connected components of S in G. Next, for a subspace $\pi \subset \mathbb{R}^n$ by $\pi_x^{\perp}$, $x \in \pi$, we denote the affine subspace of $\mathbb{R}^n$ orthogonal to π and passing through x. Then the variation $V_0(S, \pi, G)$ is defined by the formula

$$V_0(S, \pi, G) := \int_{\pi} V_0(S \cap \pi_x^{\perp}, G)\mathrm{d}x \tag{4.3}$$

where $\mathrm{d}x$ is the Lebesgue measure on π.

Further, let $\{\pi_{ij}\}$, $1 \le j \le \binom{n}{i}$, be the family of all i-dimensional coordinate subspaces in $\mathbb{R}^n$. Then the variation $V_i^*(S, G)$ of the set S in G is given by

$$V_i^*(S, G) = \frac{1}{\binom{n}{i}} \sum_{j=1}^{\binom{n}{i}} V_0(S, \pi_{ij}, G). \tag{4.4}$$

Observe that by the definition $V_n^*(S, G)$ is the Lebesgue n-measure of $S \subset \mathbb{R}^n$.

Let $Q \subset \mathbb{R}^n$ be an open parallelepiped with edges parallel to the coordinate axes. Let $S \subset\subset Q$ be a compact subset. Given $\epsilon > 0$, assume that there exist M points $y_1, \ldots, y_M \in S$ such that $\|y_i - y_j\|_\infty \ge \epsilon$. Then Vitushkin's inequality (see, e.g., [13, Chapters VII.1, VII.2]) asserts

$$M \le \sum_{j=1}^{n} \left(\frac{2}{\epsilon}\right)^j \binom{n}{j} V_j^*(S, Q). \tag{4.5}$$

4.3

Let us prove now Theorem 1.2 in case (b).

As in the proof of case (a) it suffices to establish only a local version of this theorem. Then the general result will follow from the local one using that Cov is a subadditive function in the first argument and applying arguments similar to those of Section 3.6.

So let us consider the family $F \in \mathcal{O}(U \times V)$ where $U \subset \mathbb{C}^m$ is a bounded domain and $V = B_R(0) \subset \mathbb{C}^n$ is the Euclidean ball of radius R centered at 0. Let $X_U \subset\subset U$ be a relatively compact domain. For $0 < R' < R$ let $K_{R_*} \subset B_{R'}(0) \cap \mathbb{R}^n$ be a (real) open cube of radius $R_* > 0$ centered at $0 \in \mathbb{R}^n$.

Theorem 4.2. *There exists a positive constant* $C = C(F, X_U, R', R_*)$ *such that for each* $0 < \epsilon \le 1$, $0 < t \le \epsilon$, $u \in X_U$

$$Cov\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u}); t\right) \le C \cdot \epsilon \cdot t^{-n};$$

here

$$L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u}) := \{x \in K_{R_*};\, |F(u,x)| < M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u}\}$$

and $d_u := d_F(u; B_{R'}(0))$ *is the corresponding Chebyshev degree.*

Proof. Let $K_{R''} \subset B_{R'}(0) \cap \mathbb{R}^n$ be an open cube of radius R'' centered at 0 containing the closure $\overline{K}_{R_*}$ of K_{R_*}. Let $S \subset \overline{K}_{R_*}$ be a closed sublevel set of a function $F(u, \cdot)$, $u \in X_U$. The basic result used in the proof of the theorem is:

Lemma 4.3. *There exists a positive constant $\widetilde{C} = \widetilde{C}(F, X_U, R', R_*)$ such that $V_i^*(S, K_{R''}) \le \widetilde{C}$, $1 \le i \le n-1$.*

Proof. We set

$$m := \sup_{u \in X_U, z \in B_{R'}(0)} |F(u, z)|. \tag{4.6}$$

Clearly, $m < \infty$.

Consider a real analytic function H on $U \times B_R(0) \times (-2, 2) \subset \mathbb{R}^{2m} \times \mathbb{R}^{2n} \times \mathbb{R}$ defined by

$$H(u, z, t) := |F(u, z)|^2 - t^2 \cdot m^2. \tag{4.7}$$

Next, by definition the set S is given for some $0 < c \le 1$ by the formula

$$S := \{z \in K_{R_*}; |F(u, z)| \le c \cdot M_{F(u,\cdot)}(\overline{K}_{R_*})\}.$$

Since $\gamma := \frac{M_{F(u,\cdot)}(K_{R_*})}{m} \le 1$, the set S coincides with the intersection with $\overline{K}_{R_*}$ of the semi-analytic set $S' := \{z \in B_R(0); H(u, z, \gamma \cdot c) \le 0\}$. Also, the intersection of S with the space $\pi_x^\perp$ where π_x is an i-dimensional subspace in $\mathbb{R}^n$ passing through $x \in \overline{K}_{R_*}$ and parallel to one of the i-dimensional coordinate subspaces is the same as the intersection of the semi-analytic set S' with $\pi_x^\perp$. Since $\overline{X}_U \times \overline{K}_{R_*} \times [0, 1]$ is compact, all these semi-analytic sets are compact cross-sections of the global semi-analytic set $\{(z, u, t) \in U \times B_R(0) \times (-2, 2); H(u, z, t) \le 0\}$.

Taking a finite cover of $\overline{X}_U \times \overline{K}_{R_*} \times [0, 1]$ by closed cubes in $\mathbb{R}^{2m+2n+1}$ so that each cube belongs to $U \times B_R(0) \times (-2, 2)$ and applying to each cube and the portion of the semi-analytic set S' there (which is also semi-analytic) the Gabrielov finiteness result (4.1), we obtain that each $V_0(S, \pi_{ij}, K_{R''})$ in (4.4) is bounded by a constant depending on F, X_U, R' and R_* only.

This gives the required result. □

Next, we prove:

Lemma 4.4. *The Lebesgue n-measure of $S := L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u})$ satisfies*

$$\mathcal{L}_n(S) \le 4n \cdot (2R_*)^n \cdot \epsilon.$$

Proof. According to the Yu. Brudnyi–Ganzburg type inequality (1.5)

$$M_{F(u,\cdot)}(K_{R_*}) \le \left(\frac{4n\mathcal{L}_n(K_{R_*})}{\mathcal{L}_n(S)}\right)^{d_u} \cdot M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u}.$$

This implies the required result. □

Now let us proceed with the proof of Theorem 4.2. Using Lemmas 4.3 and 4.4, and inequality (4.5) we obtain that the number of points M of an $\frac{\epsilon}{2}$-net in S where $S \subset \overline{K}_{R_*}$ is the minimal closed sublevel set of $F(u, \cdot)$ containing $L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u})$ satisfies for some $C = C(\widetilde{C}, R_*, n)$

$$M \le C \cdot \epsilon^{1-n}.$$

Since $Cov\left(L_{F(u,\cdot);R_*}(M_{F(u,\cdot)}(K_{R_*}) \cdot \epsilon^{d_u}); \epsilon\right) \le M$, this gives the required result for $t = \epsilon$. For $t \le \epsilon$ the required inequality follows directly from here as in the proof of case (a) of Theorem 1.2.

The proof of Theorem 4.2 is complete. □

As was mentioned above Theorem 4.2 implies Theorem 1.2 in case (b).

Unlike the proof of Theorem 3.1, the proof of Theorem 4.2 is nonconstructive and is based on a bound in the Gabrielov theorem. In the next example we describe a class of families F for which the constant C in Theorem 4.2 (with K_{R_*} substituted for a Euclidean ball) can be explicitly estimated. We also raise a question on the constructive proof of Theorem 4.2.

Example 4.5. Let $X \subset \mathbb{R}^n$ be a real algebraic variety of dimension $1 \le dimX \le n$ defined by equations $f_1 = 0, \dots, f_p = 0$ where f_j are real polynomials on $\mathbb{R}^n$. By $X_c \subset \mathbb{C}^n$ we denote the complex algebraic variety obtained by complexification of equations determining X.

Assume that there exist open sets $O \subset X$ and $O_c \subset X_c$ with $O \subset\subset O_c$ and a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^l$, $l := dimX$, such that $\pi|_{X_c} : O_c \to \mathbb{D}_4^l$ is biholomorphic and maps O onto $K_2^l(0) \subset \mathbb{R}^l$. In particular, O is a submanifold of X. These conditions hold, e.g., in a neighbourhood of a smooth point of $X \subset \mathbb{C}^n$.

We set $s := (\pi|_X)^{-1} : \mathbb{D}_4^l \to O_c$ and consider the family $F \in \mathcal{O}(U \times V)$ with $U := \mathbb{C}^{d_{k,n}}$ and $V := \mathbb{D}_4^l$ such that $F(u, z) := (p \circ s)(z)$ where $p \in \mathcal{P}_k(\mathbb{C}^n)$ and u is the vector of coefficients of p ordered lexicographically. (So F is a family of holomorphic algebraic functions on V.)

Let $L \subset \mathbb{R}^l$ be an affine subspace of dimension r. Then $\pi^{-1}(L) \cap \mathbb{R}^n$ is naturally identified with $\mathbb{R}^l$. Also, a closed sublevel set of a polynomial $f \in \mathcal{P}_k(\mathbb{C}^n)$ intersected with $\pi^{-1}(L \cap \overline{B}_t(0)) \cap X \subset \mathbb{R}^n$, where $\overline{B}_t(0) \subset \mathbb{R}^l$ is the closed Euclidean ball of radius $1 \le t \le 2$ centered at 0, is given by the equations

$$f_1|_{\pi^{-1}(L)\cap\mathbb{R}^n} = 0, \dots, f_p|_{\pi^{-1}(L)\cap\mathbb{R}^n} = 0, \quad (-|f|^2 + c)|_{\pi^{-1}(L)\cap\mathbb{R}^n} \ge 0 \text{ for some } c > 0,$$

$$g \circ \pi \ge 0 \quad \text{where } g(x_1, \dots, x_l) := -\sum_{j=1}^{l} x_j^2 + t^2, \quad (x_1, \dots, x_l) \in \mathbb{R}^l.$$

This and Milnor's theorem (see Example 4.1) imply that for every affine subspace $L \subset \mathbb{R}^l$ of dimension r and every $u \in U$ the number of connected components of the intersection of a closed sublevel set of $F(u, \cdot)|_{\overline{B}_t(0)}$ with L is bounded by $(1 + d + 2k + 2)^r$ where $d := degf_1 + \dots + degf_p$.

Let S be the intersection of a closed sublevel set of $F(u, \cdot)$ with $\overline{B}_1(0)$ $(\subset \overline{K}_1^l(0))$. Then (4.4) and the above bound for the number of connected components imply for every $Q := K_t^l(0)$, $1 < t \le 2$, and $1 \le i \le l - 1$

$$V_i^*(S, Q) \le (3 + d + 2k)^{n-i} \cdot 2^i.$$

We set $X_U := \mathbb{D}^{d_{k,n}}$, $X_V := B_1(0)$ and $\widetilde{X}_V := \mathbb{D}_3^l$. Then like for Lemma 4.4 we obtain from [1, Theorem 1.9] for $S := L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V) \cdot \epsilon^{d_u})$

$$L_n(S) \le 4l \cdot \mathcal{L}_n(B_1(0)) \cdot \epsilon \le 4l \cdot 2^l \cdot \epsilon.$$

Also, there exists a positive constant c depending on O and $\pi|_{O_c}$ such that for every $u \in X_u$

$$d_u := d_F(u; \widetilde{X}_V) \le c \cdot k \cdot degf_1 \cdots degf_p =: b_u$$

(see [3, Theorem 1.3]).

Combining the above inequalities with the Vitushkin inequality (4.5) as in the proof of Theorem 4.2 we obtain for $0<\epsilon\le 1$ and $k\ge 11$

$$
\begin{aligned}
&Cov\left(L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{b_u});\epsilon\right)\\
&\le\sum_{j=1}^{l-1}\left(\frac{4}{\epsilon}\right)^j\binom{l}{j}\cdot(3+d+2k)^{n-j}\cdot 2^j+\left(\frac{4}{\epsilon}\right)^{l-1}\cdot 2^{l+4}\cdot l\\
&<(11+d+2k)^l\cdot(3+d+2k)^{n-l}\cdot\epsilon^{1-l}.
\end{aligned}
$$

5. Proofs

5.1. *Proof of* Corollary 1.5

The proof follows from inequalities of Theorem 1.2 and the fact that for an open relatively compact set $S\subset\mathbb{R}^N$ such that $\overline{S}\setminus S$ has Lebesgue N-measure 0,

$$\lim_{t\to 0}t^N\cdot Cov(S;t)=\mathcal{L}_N(S).\quad\square$$

5.2. *Proof of* Corollary 1.8

If a 2ϵ-separated set S with $0<\epsilon\le 1$ belongs to X_V and $card S>C\cdot\epsilon^{2^s(1-n)}$ where C is the constant in (1.10), then $S\not\subset L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u})$ for every $u\in X_U$. In fact, if open cubes of radius ϵ cover $L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u})$, then by the definition of S each such cube contains at most one point of S. Thus if $S\subset L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u})$ then the number of cubes of such a cover is $>C\cdot\epsilon^{2^s(1-n)}$ contradicting (1.10).

Hence, there exists $s\in S$ such that $s\notin L_{F(u,\cdot)}(M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u})$. This implies

$$\sup_S|F(u,\cdot)|\ge|F(u,s)|\ge M_{F(u,\cdot)}(X_V)\cdot\epsilon^{d_u}$$

as required. $\square$

5.3. *Proof of* Theorem 2.1

Fix an open connected neighbourhood $W\subset\subset\widetilde{X}_V$ of the closure $\overline{X}_V$ of X_V and let $R>0$ be such that each open ball of radius $2R$ in $\mathbb{C}^n$ centered at a point of the closure $\overline{W}$ belongs to $\widetilde{X}_V$. For the family $F\in\mathcal{O}(U\times V)$ consider a new family $\widetilde{F}\in\mathcal{O}(U\times W\times\mathbb{D}_R\times B_2(0))$ with the parameter domain $U\times W\times\mathbb{D}_R\subset\mathbb{C}^{m+n+1}$ and the domain of functions $B_2(0)\subset\mathbb{C}^n$ defined by the formula

$$\widetilde{F}(u,w,t,z):=F(u,w+tz).\tag{5.1}$$

By the definition of W there exists $R_*<R$ such that the union of all open balls of radius $2R_*$ centered at points of $\overline{X}_V$ is a relatively compact subset of $\widetilde{X}_V$. Applying to the restriction of $\widetilde{F}$ to $X_U\times X_V\times\mathbb{D}_{R_*}\times B_1(0)$ Theorem 1.2 and then Corollary 1.8 and observing that in this case $\widetilde{F}(u,w,t,\cdot):=F(u,\cdot)|_{B_t(w)}$ and

$$\sup_{w\in X_V,t\in\mathbb{D}_R}n_{\widetilde{F}}(u,w,t;B_2(0))\le n_F(u;\widetilde{X}_V),$$

$$\sup_{w\in X_V,t\in\mathbb{D}_R}d_{\widetilde{F}}(u,w,t;B_2(0))\le d_F(u;\widetilde{X}_V)$$

we obtain that there exists a constant $C' \geq 1$ depending on F, X_U, X_V, R and R_* such that if Y is a 2ϵ-separated set in $B_r(w)$, $w \in X_V$, $r \leq R_*$, $0 < \epsilon \leq r$, such that $cardY > C' \cdot \left(\frac{\epsilon}{r}\right)^{2^s(1-n)}$ then for every $u \in X_U$

$$\sup_{z \in B_r(w)} |F(u,z)| \leq \left(\frac{\epsilon}{r}\right)^{-d_u} \cdot \sup_{z \in Y} |F(u,z)| \tag{5.2}$$

(here we retain the notation of Theorem 1.2).

Further, under the notation of Theorem 2.1 assume that $\omega \subset K_r(x; S)$ where $x \in S$, $S \in \mathcal{R}(\psi, c)$ and $r \leq \frac{R_*}{\sqrt{n}}$. By the definition $K_r(x; S) \subset B_{\sqrt{n}r}(x) \subset B_{R_*}(x)$. Therefore we can apply (5.2) in this case. Let $S \subset \omega$ be a 2ϵ-net. Then closed cubes of radius 2ϵ centered at points of S cover ω and applying to these cubes inequality (2.1) we obtain

$$\mu_\psi(\omega) \leq c \cdot cardS \cdot \psi(2\epsilon) := c \cdot cardS \cdot (2\epsilon)^{2^s(n-1)} \phi(2\epsilon).$$

Choose here

$$\epsilon := \frac{1}{2}\phi^{-1}\left(\frac{\mu_\psi(\omega)}{c \cdot C' \cdot 2^{2^s(n-1)+1} \cdot r^{2^s(n-1)}}\right).$$

Then we have $0 < \epsilon < \frac{r}{2}$ and

$$cardS > C' \cdot \left(\frac{\epsilon}{r}\right)^{2^s(1-n)}.$$

This and (5.2) imply

$$\sup_{z \in K_r(x;S)} |F(u,z)| \leq \left[\frac{1}{2r}\phi^{-1}\left(\frac{\mu_\psi(\omega)}{c \cdot \widetilde{C} \cdot r^{2^s(n-1)}}\right)\right]^{-d_u} \cdot \sup_{z \in \omega} |F(u,z)| \tag{5.3}$$

where $\widetilde{C} := C' \cdot 2^{2^s(n-1)+1}$.

This proves the result for $K_r(x; S)$ with $r \leq \frac{R_*}{\sqrt{n}}$.

Consider now $\omega \subset K_r(x; S)$ for $r > \frac{R_*}{\sqrt{n}}$. Let Y be a $\frac{R_*}{2\sqrt{n}}$-net in $K_r(x; S)$. Since $diamX_V < \infty$, there exists N depending on X_V, R_*, n such that $cardY \leq N$.

Next, the union of cubes $K_l(y; S)$ of radius $l := \frac{R_*}{\sqrt{n}}$ centered at points y of Y cover $K_r(x; S)$. Therefore there exists a point $y \in Y$ such that

$$\mu_\psi(\omega \cap K_l(y; S)) \geq \frac{\mu_\psi(\omega)}{N}.$$

Applying (5.3) to $\omega \cap K_l(y; S) \subset K_l(y; S)$ we obtain

$$\sup_{z \in K_l(y;S)} |F(u,z)| \leq \left[\frac{1}{2r}\phi^{-1}\left(\frac{\mu_\psi(\omega)}{c \cdot \widetilde{C} \cdot N \cdot r^{2^s(n-1)}}\right)\right]^{-d_u} \cdot \sup_{z \in \omega} |F(u,z)|. \tag{5.4}$$

Finally, according to the Yu. Brudnyi–Ganzburg type inequality (1.5) we have for some constant C depending on X_U, $\widetilde{X}_V$, F, R_* and $u \in X_U$

$$\sup_{K_r(x;S)} |F(u,\cdot)| \leq \sup_S |F(u,\cdot)| \leq C \cdot \sup_{z \in K_l(y;S)} |F(u,z)|. \tag{5.5}$$

Combining inequalities (5.4), (5.5) completes the proof of Theorem 2.1. $\square$

5.4. *Proof of* Corollary 2.2

Let us recall the definition of *rearrangement*. Let (Σ, μ) be a measure space and $f : \Sigma \to \mathbb{R}$ be μ-measurable. A nonincreasing function $m(f) : (0, \infty) \to \mathbb{R}_+ \cup \{\infty\}$ is then given by

$$m(f; t) := \mu\{\sigma \in \Sigma; |f(\sigma)| > t\},$$

while the rearrangement $f^* : (0, \mu(\Sigma)] \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$f^*(s) := \inf\{t; m(f; t) \le s\}.$$

Functions $|f|$ and f^* are *equimeasurable*; therefore, for a measurable function $\Phi : [0, \mu(\Sigma)] \to \mathbb{R}$

$$\int_0^{\mu(\Sigma)} \Phi(f^*(s))\mathrm{d}s = \int_\Sigma \Phi(|f|)\mathrm{d}\mu.$$

Under the notation of Theorem 2.1, assuming that $\sup_{z \in K_r(x;S)} |F(u, z)| \neq 0$ consider

$$g(u, z) := \frac{F(u, z)}{\sup\limits_{z \in K_r(x;S)} |F(u, z)|}, \quad z \in K_r(x; S).$$

For $t \in (0, 1]$ we set

$$S_{g(u,\cdot)}(t) := \{z \in K_r(x; S); |g(u, z)| \le t\}.$$

Then from inequality (2.2) we obtain

$$\mu_\psi(S_{g(u,\cdot)}(t)) \le C_2 \cdot c \cdot r^{2^s(n-1)} \cdot \phi\left(\frac{2 \cdot r \cdot t^{1/d_u}}{C_1}\right).$$

Next, by the definition

$$\mu_\psi(S_{g(u,\cdot)}(t)) = \mu_\psi(K_r(x; S)) - m(g(u, \cdot); t).$$

Also, the inverse to the function $t \mapsto \mu_\psi(S_{g(u,\cdot)}(t))$ is $t \mapsto (g(u, \cdot))^*(\mu_\psi(K_r(x; S)) - t)$. Thus, the above inequality implies

$$(g(u, \cdot))^*(\mu_\psi(K_r(x; S)) - t) \ge \left[\frac{C_1}{2r}\phi^{-1}\left(\frac{t}{C_2 \cdot c \cdot r^{2^s(n-1)}}\right)\right]^{d_u}. \tag{5.6}$$

Therefore

$$\begin{aligned}
&\int_0^1 \left[\frac{C_1}{2r}\phi^{-1}\left(\frac{\mu_\psi(K_r(x; S)) \cdot t}{C_2 \cdot c \cdot r^{2^s(n-1)}}\right)\right]^{d_u} \mathrm{d}t \\
&= \frac{1}{\mu_\psi(K_r(x; S))} \int_0^{\mu_\psi(K_r(x;S))} \left[\frac{C_1}{2r}\phi^{-1}\left(\frac{t}{C_2 \cdot c \cdot r^{2^s(n-1)}}\right)\right]^{d_u} \mathrm{d}t \\
&\le \frac{1}{\mu_\psi(K_r(x; S))} \int_{K_r(x;S)} |g(u, \cdot)|\mathrm{d}\mu_\psi.
\end{aligned}$$

This gives inequality (1).

Inequality (2) is obtained similarly using (5.6). □

Corollary 2.3 is proved in the same way.

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